NASA TT F-10639

HYPERSONIC PROFILE OF MINIMUM DRAG, HAVING AN ASSIGNED BENDING STRENGTH

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Translation of "Giperzvukovoy profil', imeyushchiy zadannuyu prochnost' na izgib i minimal'noye soprotivleniye".
Izvestiya Vysshikh Uchebnykh Zavedeniy, Seriya Aviatsionnaya Tekhnika, No. 2, pp. 119-125, 1966.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON D.C. JANUARY 1967

HYPERSONIC PROFILE OF MINIMUM DRAG, HAVING AN ASSIGNED BENDING STRENGTH

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The authors try to find the most advantageous form of a hypersonic profile which, for a given drag torque, has minimum drag. The problem is solved by a variational method.

In order to enhance the aerodynamic quality of the supporting surfaces /119*
of hypersonic aircrafts, it is necessary to diminish the thickness of the profile, which in turn leads to a decrease in the strength of the airfoil. By a favorable distribution of masses, one can increase the drag torque, and thereby raise the bending strength.

In the present work, we have tried to find the most advantageous form of a hypersonic profile which, for a given drag torque, has minimum drag. The problem is solved by a variational method. The pressure coefficient is determined from Newton's formula (Ref. 1)

$$\bar{p} = 2\sin^2\nu, \tag{1}$$

where ν is the angle between the direction of the flow and the tangent to the surface of the aircraft.

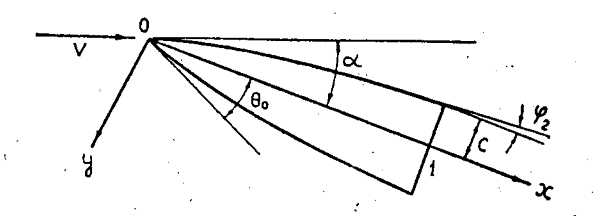

Figure 1

Let $y = y(x)$ be the equation of the hypersonic profile. Then $\nu = \alpha + \theta$, where $\tan\theta = y'$ (Figure 1).

For thin profiles, operating at a small angle of attack, we may assume:

$$\sin\nu = \alpha + \theta, \quad \mathrm{tg}\,\theta = \theta = \bar{c}\,\frac{d\bar{y}}{d\bar{x}}.$$

Here $\bar{x} = \frac{x}{b}$, $\bar{y}$** $= \frac{y}{c}$, $\bar{c} = \frac{b}{c}$ is the relative thickness of the profile, b is the chord.

The pressure coefficient in this case has the form:

$$\bar{p} = 2(\alpha + \bar{c}\,\bar{y}')^2. \tag{1)*$$

If $\alpha > \theta_0$, then the upper surface will lie in the aerodynamic shadow, and the pressure coefficient at it will equal zero ($\bar{p}_u = 0$).

* Numbers in the margin indicate pagination in the original foreign text.

** Translator's note: The original Russian gives $\bar{v}$, but it should clearly be $\bar{y}$.

The lift and drag coefficients are determined by the form of the lower surface: /120

$$C_y = 2\int_0 (\alpha + \bar{c}\,\bar{y}')^2\, d\bar{x}, \tag{2}$$

$$C_x = 2\int_0 (\alpha + \bar{c}\,\bar{y}')^3\, d\bar{x}. \tag{3}$$

Bending strength torque will be written as

$$W = \frac{I}{r},$$

where I is the moment of inertia referred to the neutral axis,
r is the distance between the neutral axis and the most remote point of the surface of the profile.

The location of the neutral axis depends, in general, on the form of the profile. If the profile is symmetrical, then the neutral axis coincides with the axis of symmetry x. In this case, for a solid profile the moment of inertia will be

$$I_S = \frac{\bar{c}^3 b^4}{12}\int_0 \bar{y}^3\, d\bar{x}, \tag{4}$$

for a non-solid profile with a small wall thickness δ it will be

$$I_T = 2\delta\bar{c}^2 b^3 \int_0 \bar{y}^2\, d\bar{x}. \tag{4a}$$

Correspondingly, the bending strength torque is:

$$W_S = \frac{\bar{c}^2 b^3}{6}\int_0 \bar{y}^3\, d\bar{x} = A\bar{c}^2 \int_0 \bar{y}^3\, d\bar{x} - \text{for the solid profile,} \tag{5}$$

$$W_T = 4\delta\bar{c} b^2 \int_0 \bar{y}^2\, d\bar{x} = A_T\,\bar{c}\int_0 \bar{y}^2\, d\bar{x} - \text{the hollow profile.} \tag{5a}$$

The determination of a profile of minimum drag for a given lifting power and bending strength torque amounts to finding an extremum at which the functional (3) attains its minimum for pre-assigned values of the functionals (2), (5a) or (5). To solve this problem by Lagrange's method, we shall construct a functional having F as the integrand:

$$F = (\alpha + \bar{c}\,\bar{y}')^3 + \lambda_0(\alpha + \bar{c}\,\bar{y}')^2 + \lambda_1 \bar{y}^n. \tag{6}$$

Here n = 3 for the solid profile, and n = 2 for the thin-walled profile. Euler's equation will have the form:

$$\lambda_1 n\bar{y}^{n-1} - \bar{c}\frac{d}{d\bar{x}}\{3(\alpha + \bar{c}\,\bar{y}')^2 + 2\lambda_0(\alpha + \bar{c}\,\bar{y}')\} = 0. \tag{7}$$

Since it does not contain an explicitly independent variable, then -- introducing the new function

$$\varphi = \bar{y}', \tag{8}$$

we obtain a first-order equation:

$$\lambda_1 n\bar{y}^{n-1} - \bar{c}\varphi \frac{d}{d\bar{y}} \{3(\bar{c}\varphi + \alpha)^2 + 2\lambda_0(\alpha + \bar{c}\varphi)\} = 0. \tag{9}$$

If $\lambda_1 = 0$, then (7) has the solution $\bar{\nu}' = \text{const}$. For $\lambda_1 \neq 0$, the variables in (9) are separated:

$$\bar{y} = \tilde{\lambda}_1 \sqrt[n]{\varphi^3 + \tilde{\lambda}_0\varphi^2 + C_1}. \tag{10}$$

Here $\tilde{\lambda}_1$, $\tilde{\lambda}_0$ are new arbitrary coefficients. If one considers ϕ as a parameter, then we shall obtain the independence of $\bar{x}$ on the parameter by making use of the relations:

$$d\bar{x} = \frac{d\bar{y}}{\varphi}, \tag{11}$$

$$\bar{x} = \frac{\tilde{\lambda}_1}{n} \int (C_1 + \tilde{\lambda}_0\varphi^2 + \varphi^3)^{\frac{1-n}{n}} (2\tilde{\lambda}_0 + 3\varphi)\, d\varphi + C_2. \tag{12}$$

We substitute (12) in the expressions (2), (3), (5) or (5a). Taking ϕ as an independent variable we find:

$$C_y = 2\bar{c}^2 \frac{\tilde{\lambda}_1}{n} \int_{\varphi_0}^{\varphi_2} \left(\frac{\alpha}{\bar{c}} + \varphi\right)^2 (C_1 + \tilde{\lambda}_0\varphi^2 + \varphi^3)^{\frac{1-n}{n}} (2\tilde{\lambda}_0 + 3\varphi)\, d\varphi, \tag{13}$$

$$C_x = 2\bar{c}^3 \frac{\tilde{\lambda}_1}{n} \int_{\varphi_0}^{\varphi_2} \left(\frac{\alpha}{\bar{c}} + \varphi\right)^3 (C_1 + \tilde{\lambda}_0\varphi^2 + \varphi^3)^{\frac{1-n}{n}} (2\tilde{\lambda}_0 + 3\varphi)\, d\varphi, \tag{14}$$

$$W = A\bar{c}^{n-1} \frac{\tilde{\lambda}_1^{n+1}}{n} \int_{\varphi_0}^{\varphi_2} (C_1 + \tilde{\lambda}_0\varphi^2 + \varphi^3)^{\frac{1}{n}} (2\tilde{\lambda}_0 + 3\varphi)\, d\varphi. \tag{15}$$

The limits of integration ϕ_0 and ϕ_2 are calculated using the initial conditions since ϕ_0 and ϕ_2 replace c_1 and c_2:

(1) $\phi = \phi_0$ for $\bar{x} = 0$, $\bar{y} = 0$,

(2) $\phi = \phi_2$ for $\bar{x} = 1$, $\bar{y} = 1$.

From the first condition we have:

$$C_1 = -\varphi_0^3 - \tilde{\lambda}_0\varphi_0^2.$$

The parametric equation of the profile will be:

$$\bar{x} = \frac{\tilde{\lambda}_1}{n} \int_{\varphi_0}^{\varphi} [\varphi^3 - \varphi_0^3 + \tilde{\lambda}_0(\varphi^2 - \varphi_0^2)]^{\frac{1-n}{n}} (2\tilde{\lambda}_0 + 3\varphi)\, d\varphi,$$
$$\bar{y} = \tilde{\lambda}_1 \sqrt[n]{(\varphi^3 - \varphi_0^3) + \tilde{\lambda}_0(\varphi^2 - \varphi_0^2)}. \tag{16}$$

The integrals (13)-(16) can be evaluated in terms of elementary functions for $n = 3$ and $\alpha = \lambda_0$, i.e., if we set ourselves a problem of a profile of minimum drag having a solid cross-section for a given drag torque, lifting power being not pre-assigned.

In this case:

$$\bar{y}=\tilde{\lambda}\sqrt[3]{\varphi^3-\varphi_0^3}, \tag{17}$$

$$\bar{x}=\tilde{\lambda}\int_{\varphi_0}^{\varphi}\frac{\varphi d\varphi}{\sqrt[3]{(\varphi^3-\varphi_0^3)^2}}. \tag{18}$$

The integral (18), defining $\bar{x}$, is a binomial differential. Performing calculations with the aid of the substitution

$$t=\sqrt[3]{1-\left(\frac{\varphi_0}{\varphi}\right)^3}=\sqrt[3]{1-\frac{1}{\mu^3}},$$

we shall obtain the dependence of $\bar{x}$ on the parameter μ, expressed in terms of elementary functions

$$\bar{x}=\tilde{\lambda}\left[\frac{1}{\sqrt{3}}\operatorname{arc\,tg}\frac{2\sqrt[3]{\mu^3-1}+\mu}{\sqrt{3}\mu}-\frac{\pi}{6\sqrt{3}}-\right.$$
$$\left.-\frac{1}{2\cdot 3}\ln\frac{(\mu-\sqrt[3]{\mu^3-1})^2}{\mu^2+\mu\sqrt[3]{\mu^3-1}+\sqrt[3]{(\mu^3-1)^2}}\right]. \tag{19}$$

From (2) with the aid of (19) we find $\tilde{\lambda}$ (setting $\mu = \mu_2$ for $\bar{x} = 1$). Then from (17), making use of the condition $\bar{y} = 1$ for $\mu = \mu_2$, we find ϕ_0. For example, for $\mu_2 = 0$ we find that the integral in (18) is equal to $\frac{2\pi}{3\sqrt{3}} = 1.212$, and, consequently, $\lambda = -0.825$ and $\phi_0 = 1.212$. Assigning other values to μ_2, we determine the corresponding $\tilde{\lambda}$ and ϕ_0.

With the help of the same substitution as in (18), the integral determining C_x is calculated for a solid profile:

$$C_x=2\bar{c}^3\tilde{\lambda}\varphi_0^3\int_1^{\mu_2}\frac{\mu^4}{\sqrt[3]{(\mu^3-1)^2}}d\mu=$$
$$=2\bar{c}^3\tilde{\lambda}\varphi_0^3\frac{1}{3}\left[\mu_2^2\sqrt[3]{\mu_2^3-1}+\frac{2}{\sqrt{3}}\operatorname{arc\,tg}\frac{2\sqrt[3]{\mu_2^3-1}+\mu_2}{\sqrt{3}\mu_2}-\right.$$
$$\left.-\frac{1}{3}\ln\frac{(\mu_2-\sqrt[3]{\mu_2^3-1})^2}{\mu_2^2+\mu_2\sqrt[3]{\mu_2^3-1}+\sqrt[3]{(\mu_2^3-1)^2}}-\frac{2\pi}{\sqrt{3}\cdot 6}\right]. \tag{20}$$

If $\mu_2 = 0$, then $C_x = 1.19\cdot 2\bar{c}^3$, i.e., the drag of the profile obtained is 19% higher than the drag of a wedge with the same relative thickness $\bar{c}$.

Let us find the bending strength torque of the optimum profile substituting

$\bar{y}$ and $\bar{x}$ expressed in terms of ϕ in formula (5):

$$W = A_S\bar{c}^2\tilde{\lambda}^4 \int_{\varphi_0}^{\varphi_2} \sqrt[3]{\varphi^3 - \varphi_0^3} \cdot d\varphi,$$

$$W = A_S\bar{c}^2\tilde{\lambda}^4\varphi_0^3\frac{1}{3}\left[\mu_2^2\sqrt[3]{\mu_2^3-1} + \frac{1}{6}\ln\frac{(\mu_2 - \sqrt[3]{\mu_2^3-1})^2}{\mu_2^2 + \mu_2\sqrt[3]{\mu_2^3-1} + \sqrt[3]{(\mu_2^3-1)^2}} - \right.$$
$$\left. - \frac{1}{\sqrt{3}}\operatorname{arc\,tg}\frac{2\sqrt{\mu_2^3-1}+\mu_2}{\sqrt{3}\mu_2} + \frac{\pi}{6\sqrt{3}}\right]. \tag{21}$$

For $\mu_2 = 0$ we obtain $W = A_s\bar{c}^2 \cdot \frac{1}{3}$. For a wedge-shaped solid profile of the /123
same thickness, the bending strength torque will be 25% lower:

$$W_W = A_c\bar{c}^2 \cdot \frac{1}{4}.$$

To increase the strength, the thickness of the wedge-shaped profile should be larger:

$$\bar{c}_W = \sqrt{\frac{4}{3}}\bar{c}_{opt}. \tag{22}$$

The drag of a wedge with the same strength is

$$C_{xW} = 2\bar{c}_W^3 = 1{,}15^3 \cdot 2\bar{c}_{opt}^3$$

and is 29% higher than that for the optimum profile, since $\frac{C_{xW}}{C_{x\ opt}} = \frac{1.15^2}{1.19} =$ 1.29. From (15) and (16), setting $\phi = \phi_2$ or $\mu = \mu_2$, we obtain the conditions for the determination of $\tilde{\lambda}_1$ and ϕ_0:

$$\tilde{\lambda}_1 \varphi_0^{\frac{3}{n}} R_y = 1, \tag{23}$$

$$\frac{\tilde{\lambda}_1}{n} \cdot \varphi^{\frac{3}{n}-1} \cdot R_x = 1. \tag{24}$$

Here

$$R_y = \sqrt[n]{1 - \mu_2^3}, \tag{25}$$

$$R_x = \int_1^{\mu_2} 3\mu(1-\mu^3)^{\frac{1}{n}-1} \cdot d\mu. \tag{26}$$

Hence

$$\varphi_0 = \frac{R_x}{nR_y}$$

and

$$\tilde{\lambda} = \left(\frac{R_y n}{R_x}\right)^{\frac{3}{n}} \cdot R_y.$$

The drag and the bending strength torque can be similarly represented

$$C_x = 2\bar{c}^3 \frac{\tilde{\lambda}}{n} \varphi_0^{\frac{3}{n}+2} I_x = 2\bar{c}^3 \frac{I_x}{n^3} \frac{R_x^2}{R_y^3}, \tag{27}$$

$$W = A\bar{c}^{n-1} \frac{\tilde{\lambda}^{n+1}}{n} \varphi_0^{\frac{3}{n}+2} I_w = A\bar{c}^{n-1} \frac{I_w}{R_x R_y^n}, \tag{28}$$

where

$$I_x = \int_1^{\mu_2} 3\mu^4 (1-\mu^3)^{\frac{1}{n}-1} d\mu, \tag{29}$$

$$I_w = \int_1^{\mu_2} 3\mu (1-\mu^3)^{\frac{1}{n}} d\mu. \tag{30}$$

Substituting these expressions in the conditions of equality of bending /124 strength torques, we obtain the relations between the thicknesses of the optimum profiles and the wedge-shaped profiles:

$$\bar{c}_W = \bar{c}_{\text{prof.}} \sqrt[n-1]{\frac{I_w(n+1)}{R_x R_y^n}}. \tag{31}$$

The ratio of drags for these profiles is proportional to the cube of the ratio of thicknesses:

$$\frac{C_{xW}}{C_{x\ \text{prof.}}} = \left(\frac{\bar{c}_W}{\bar{c}_{\text{prof.}}}\right)^3 \frac{R_x}{I_x} \frac{1}{\varphi_0^3} = \left[\frac{(n+1) I_w}{R_y}\right]^{\frac{3}{n-1}} \frac{n^3}{I_x R_x^{\frac{2n+1}{n-1}}}. \tag{32}$$

The dependence of R_x, I_w, I_x, and ϕ_0 for various values of μ_2 for a solid profile ($n = 3$) is given in the table. As may be seen, all the quantities depend only slightly on ϕ_2. For a thin-walled profile ($n = 2$), these integrals can be calculated only approximately.

The integrals which determine $\bar{x}$ and C_x are improper integrals, and for $\phi = \phi_0$ (or $\mu = 1$) the integrand has a discontinuity. This singularity prevents us from calculating the integrals by the usual methods of numerical integration for the entire interval at once. We shall divide the interval of integration and the corresponding integral into two parts. The first part -- with a discontinuity, the second -- with a continouous integrand:

$$I = \int_1^{1-\varepsilon} f(x)\varphi(x)\,dx + \int_{1-\varepsilon}^{\mu_2} f(x)\varphi(x)\,dx.$$

The second term is calculated by the usual methods of numerical integration.

In the neighborhood of the point of singularity, we can apply the mean-value theorem

$$\int_1^{1-\varepsilon} f(x)\varphi(x)\,dx = \varphi(\xi) \int_1^{1-\varepsilon} f(x)\,dx,$$

where ξ denotes some intermediate point in the interval $[1, 1-\varepsilon]$.

If we take $\varepsilon = 0.1$ and $\xi = 0.95$ as the inner point of the interval, we then have

$$\int_1^{0,9} \frac{\mu d\mu}{\sqrt{1-\mu^3}} = \left(\frac{\mu}{\sqrt{\mu^2+\mu+1}}\right)_{\mu=0,95} \cdot 2\sqrt{1-\mu}\,\Big|_1^{0,9} = 0,3478.$$

The same integral for C_x will be

$$\int_1^{0,9} \frac{\mu^4 d\mu}{\sqrt{1-\mu^3}} = 0,253.$$

For the case of a thin-walled profile (n = 2), the ratio of the drags for the wedge-shaped profile and the optimum profile with the same bending strength can be written as:

$$\frac{C_{xw}}{C_{x\ prof.}} = \frac{2^3}{I_x}\left[\frac{3I_w}{R_x R_y}\right]^3 \frac{1}{R_x^2}. \qquad (33)$$

Considering that for $\mu_2 = 0$, $R_y = 1$, $R_x = 0.838$, $I_w = 3 \cdot 0.365$ and $I_x = 3 \cdot 0.408$, we obtain $\frac{C_{x\ w}}{C_{x\ opt}} = 2.27$ -- i.e., the gain in the drag for the wedge with equal bending strength over that for the optimum profile is larger than 2.

A change in the drag for other μ_2 is small, and can be estimated on the /125 basis of formula (33) by changing all the integrals:

$$\Delta\left(\frac{C_{xw}}{C_{x\,opt}}\right) = \frac{C_{xw}}{C_{x\,opt}}\left(-\frac{\Delta I_x}{I_x} + 3\frac{\Delta I_w}{I_w} - 5\frac{\Delta R_x}{R_x} - 3\frac{\Delta R_y}{R_y}\right).$$

Since, by virtue of (25), (26), (29) and (30), all the derivatives are

$$\left|\frac{dI_x}{d\mu}\right|_{\mu_2=0} = \left|\frac{dI_w}{d\mu}\right|_{\mu_2=0} = \left|\frac{dR_x}{d\mu}\right|_{\mu_2=0} = 0,$$

then ΔI and ΔR, the increments of all the integrals, will be variables of second order with respect to $\Delta_{\mu 2}$.

TABLE

$n = 3$

μ_2	0	0,05	0,10	0,20	0,30
R_x	1,212	1,180	1,176	1,122	1,074
R_y	1	1,000	1,000	0,997	0,991
I_w	1,212	1,181	1,177	1,124	1,078
I_x	2,424	2,424	2,423	2,422	2,420

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Received
February 8, 1965

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